# On Ordering of Operators in Canonical Quantization in Curved Space

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**Abstract**

Ambiguities arising in different approaches (canonical, quasiclassical, path integrals) to quantization are discussed by an example of the mechanics of a point-like particle in the Riemannian space. A way to select a single rule of quantization is proposed by requiring of consistency of the quantum mechanics' following from the canonical and quasiclassical approaches. This rule selects also a unique definition of the path integration. A geometric interpretation of noncovariance of the canonical Hamilton operator with respect to diffeomorphisms of the configuration space is proposed.

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# 1 Introduction

Various procedures of quantization (canonical, path integration, quasiclassical, deformational) of classical mechanics are more or less well–developed mathematical theories. However, being used to construct the quantum mechanics (QM)[1] for a concrete classical system, the procedures occur to be ambiguous. For example, the result of canonical quantization depends generally on the definition of ordering of operators of observables in their products, see, e.g., [1], though there is no ambiguity for the standard oscillator–like systems. This ambiguity manifests itself also in the path integral formalism, see, e.g., [2]. This situation does not trouble apparently the mathematicians, because they consider different versions of the same approach, as cohomologically equivalent [3]. However, the ambiguities are essential in physics, because they can lead, for example, to different Hamilton operators and, thus, are unequivalent from a physicist's point of view.

A point-like, chargless and spinless particle moving along a geodesic of the $n$–dimensional Riemannian space $V_n$ is an especially interesting example of a system, to which different QMs correspond under canonical quantization. This is the simplest and, therefore, elementary physical

object, and its dynamics has a clear geometrical meaning. At the same time, there are a lot of more complicated conceptual systems whose dynamics can be modelled as a motion in some $V_n$ (see, e.g. [4]).

In the present paper, it is shown that the ambiguity of ordering of operators can be removed if the condition of consistency (to an extent that can be reached) between the canonical and quasiclassical approaches to QM of the elementary particle in $V_n$ is imposed. At first sight, it looks strange to deduce a general rule of ordering

from the consideration of a concrete system, and, moreover, from the consideration of a particular operator of an observable, namely, the Hamilton one. However, if the rule is supposed to be universal, and only a single version is provided by the condition of the consistency in the particular but fundamental and geometrically determined case, then it is natural to accept the selected rule as the general one. After that, it selects "the physically true" definition of the path integral.

The paper is organized as follows. In Sec.2, the form of the Hamilton operator originally obtained by B.DeWitt [5] from the quasiclassical (WKB) propagator is discussed. The general scheme of canonical quantization and its particular versions determined by the Weyl and Rivier orderings are recalled in Sec.3. In Sec.4, the Hamilton operators in QM in $V_n$ for the mentioned orderings are written out and compared with the DeWitt's one; it is shown here that the consistency can be achieved only for a single combination of the Weyl and Rivier orderings. This

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[1]I will use the term 'quantum mechanics' for the restricted problem of quantization of the minimal set of observables that is necessary to describe the dynamics of a physical system, while 'quantization' will mean a more general problem of mapping from a sublalgebra of the Poisson algebra of functions on a phase space onto a set of operators on a Hilbert space, or for modifications of this problem.

combination is proposed as a distiguished rule of canonical quantization in general. Relation of the selection of the rule of ordering to definition of the path integral for propagator is discussed in Sec.5. Short conclusions are made in Sec.6.

## 2   Quasiclassical quantum potential

Let $\omega_{ij}(\xi)$, $i, j, k, ... = 1, ..., n$, $\{\xi^i\} \in V_n$, is the metric tensor of $V_n$, the latter being supposed an elementary manifold. Then the representation space (the space of states) can be taken as $L^2(V_n; \mathbb{C}; \sqrt{\omega}\, d^n\xi)$. The general form of a one-particle Hamilton operator in the Schrödinger representation is the same for the canonical and quasiclassical approaches[2]:

$$\hat{H}_0 \;=\; \Delta(\xi) + \; V_q(\xi) \cdot \hat{\mathbf{1}}, \quad \xi \in V_n, \tag{1}$$

where $\Delta(\xi)$ is the Laplace–Beltrami operator in $V_n$ and $V_q(\xi)$ is the so called quantum potential.

In the quasiclassical approach, DeWitt [5] came to the Hamilton operator (1) with

$$V_q^{(\mathrm{DW})}(\xi) = -\frac{\hbar^2}{2m}\frac{1}{6}R(\xi), \tag{2}$$

instead of the expected $V_q(\xi) = 0$; here $R(\xi)$ is the scalar curvature .

DeWitt [5] started with the following conjecture on a one-particle propagator[3]:

$$< \xi', t'|\xi, t > = \omega^{-1/4}(\xi')D^{1/2}(\xi'|\xi)\omega^{-1/4}(\xi) \; \exp\left(-\frac{\mathrm{i}}{\hbar}S(\xi', t'|\xi, t)\right) \tag{3}$$

where $D$ is the Van Vleck determinant, and $S(\xi', t'|\xi, t)$ is a solution of the Hamilton–Jacobi equation in both the sets of arguments. This propagator is a generalization to $V_n$ of the WKB-propagator constructed by Pauli [6] for a particle in an electromagnetic field in the flat space. Considering the limit $t' \to t$ ($\xi' \to \xi$) *along the geodesic line*

*connecting* $\xi'$ *and* $\xi$, DeWitt comes to the following  Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t'} < \xi'|\xi > + \; \frac{\hbar^2}{2m}\left(\Delta(\xi') + \frac{1}{6}R(\xi')\right) < \xi'|\xi >= o(\xi' - \xi)) < \xi'|\xi > \tag{4}$$

and, thus, to eq.(2).

This was an outstanding result that surprised DeWitt himself, because it was for the first time when the curvature appeared explicitly in a fundamental equation, thus breaking the (weak) Principle of Equivalence. (A clear exposition of the Principle can be found in [7]). Since

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[2]The index 0 is attached to $\hat{H}_0$ for consistency with notation in [10, 12] where it denotes a nonrelativistic Hamilton operator. In the present context, relativistic and nonrelativistic classical theories are equivalent and the corresponding quantum ones can differ (This is one more problem of quantization.) Thus, QM is considered here on a nonrelativistic level, see also [12]

[3] Here, I change slightly the notation in eq.(7.8) in [5] and restrict the consideration to the time–independent metric tensor $\omega_{ij}(\xi)$

beginnings of QM there was a confidence that $V_q(\xi) = 0$, based on the work by Podolski [8] on QM in the curvilinear coordinates in the Euclidean space. Much later, Sniatycki [9] came to the same result using the Blattner-Kostant-Sternberg kernel in the framework of geometric quantization. At last, for $n = 3$, there is a support in favor of $V_q^{(\mathrm{DW})}(\xi)$ from the relativistic theory of scalar field: for the space-time $T \bigotimes V_3$ eq.(1), is a non-relativistic consequence of the quantum theory of a scalar field coupled conformally to the geometry, (see details in [10]).

Consider now the general scheme of the canonical formalizm.

# 3    Canonical quantization in $V_n$

We start with the phase space $X_{2n} = R^* \bigotimes V_n$, the trivial cotangent bundle over the configurational space $V_n$, in which the Darboux (canonically conjugate) coordinates

$$\{p_i\} \in R_n^*, \quad \{\xi^i\} \in V_n \tag{5}$$

are introduced. We assume also that all coordinate lines for $\{\xi^i\}$ are complete and not closed, that is $-\infty < \xi^i < \infty$. In this sense, $\xi^i\}$ are similar to the Cartesian coordinates; this assumption is necessary because, in the canonical quantization, one starts usually with polynomials as observables and genereralize the class of functions by use of the Fourier transforms [1]. A physical interpretaion of this very restrictive assumption is that local manifestations of the space curvature are taken into account.

For our purposes, *quantization* (see, e.g., [11]) can be defined as a map

$$\mathcal{Q} : \ \mathcal{F}_{2n} \ni f(p, \xi) \longrightarrow \hat{f} \ (\text{an operator acting in } \mathcal{H} \equiv L^2(V_n; \ \mathbb{C}; \ \sqrt{\omega} \ d^n \xi)), \tag{6}$$

where $\mathcal{F}_{2n}$ is an appropriate subalgebra of the Poisson algebra of $C^\infty(X_{2n})$–functions; the map Q is assumed to satisfy the following postulates of quantization:

1)   $1 \rightarrow \hat{\mathbf{1}}$ ;

2)   $\{f, g\} \rightarrow i\hbar^{-1}(\hat{f}\hat{g} - \hat{g}\hat{f}) \overset{def}{=} i\hbar^{-1}[\hat{f}, \hat{g}]$     where $\{.,.\}$ is the Poisson bracket on $X_{2n}$ ;

3)   $\overline{\hat{f}} = (\hat{f})^\dagger$ where the overline and the dagger denote, respectively, the complex conjugation and the Hermitean conjugation with respect to the inner product in $\mathcal{H}$ ;

4)   for a complete set of functions $f^1(x), ..., f^n(x), \ f^i \in \mathcal{F}(X_{2n}), \ (\{x\} \sim \{p, \xi\}) \in X_{2n})$, the corresponding operators $\hat{f}^1, ..., \hat{f}^n$ also form a complete set, that is, if $\left[\hat{f}, \ \hat{f}^i\right] = 0$ for any $i = 1, ..., n$, then
$\hat{f} = \hat{f}(\hat{f}^1, ..., \hat{f}^n)$ .

The real functions $f(x) \in \mathcal{F}(X_{2n})$ are observables for a classical system on $X_{2n}$. The Darboux coordinates are particular cases of such functions, and, since any $f(x)$ can be expressed through these coordinates, the latter can be called basic classical observables. The

corresponding quantum operators of the observables acting in $L^2(V_n; \mathbb{C}; \sqrt{\omega} \, d^n\xi)$ are [5]

$$\xi^i \xrightarrow{\mathcal{Q}} \hat{\xi}^i = \xi^i \cdot \hat{1}, \quad p_j \xrightarrow{\mathcal{Q}} \hat{p}_j = -\mathrm{i}\hbar(\partial_j + \frac{1}{4}\partial_j \ln \omega) \tag{7}$$

The main problem is further to construct the map $f(p,\xi) \xrightarrow{\mathcal{Q}} \hat{f}$ in terms of the basic observables $\hat{p}$, $\hat{\xi}^i$. In the canonical approach, one proceeds as follows [1]:

1) Starts with (real) polynomial observables:

$$f(p,\xi) = f + \sum_{a,b} f_{i_1...i_k}^{j_1...j_l} \, \xi^{i_1}...\xi^{i_k} \, p_{j_1}...p_{j_l}; \tag{8}$$

2) substitutes $p_j$, $\xi^i$ by $\hat{p}_j$, $\hat{\xi}^i$, respectively;
3) hermitizes the obtained operator by a certain rule of ordering of $\hat{p}_j$, $\hat{\xi}^i$;
4) generalizes the rule, if possible, to a class of observables $f(p,\xi)$ wider, than that of polynomials.

There are infinitely many realizations of this scheme. They are classified, e.g., in [14]. Consider now two examples, one of which, the Weyl ordering, is the most popular one, and the both of them are important for our further consideration.

*Example 1.* According to [1], the Weyl ordering as applied to polynomial (8) consists in complete symmetrization of operators $\hat{p}$ and $\hat{q}$ in each monomial in the right-hand side of eq.(8) after substitutions $p_i \to \hat{p}_i$, $q^j \to \hat{q}^i$. The result can be represented in $V_n$ as

$$
\begin{aligned}
f(p,\xi) &\xrightarrow{\mathcal{Q}} (\hat{f}\psi)(\xi) \\
&= (2\pi\hbar)^{-n}\omega^{-\frac{1}{4}}(\xi) \int d^n\tilde{\xi} \, d^n p \, \exp\left(-\frac{\mathrm{i}}{\hbar}(\xi^i - \tilde{\xi}^i)p_i\right) f\left(p, \frac{\xi+\tilde{\xi}}{2}\right) \omega^{\frac{1}{4}}(\tilde{\xi}) \, \psi(\tilde{\xi})
\end{aligned} \tag{9}
$$

Further, this result is taken as a rule determining an operator $\hat{f}$ for any function $f(p,\xi)$ for which the expression has a meaning.

*Example 2.* Rivier [13] proposed a correspondence that is equivalent to the following one

$$
\begin{aligned}
f(p,\xi) &\xrightarrow{\mathcal{Q}} (\hat{f}\psi)(\xi) \\
&= (2\pi\hbar)^{-n}\omega^{-\frac{1}{4}}(\xi) \int d^n\tilde{\xi} \, d^n p \, \exp\left(-\frac{\mathrm{i}}{\hbar}(\xi^i - \tilde{\xi}^i)p_i\right) \frac{f(p,\xi) + f(p,\tilde{\xi})}{2} \omega^{\frac{1}{4}}(\tilde{\xi}) \, \psi(\tilde{\xi}).
\end{aligned} \tag{10}
$$

An argument in favor of this rule was that it provides one-to-one correspondence between infinitesimal canonical transformations on $X_{2n}$ and infinitesimal unitary transformations in $\mathcal{H}$. It is easily shown that the Rivier rule of quantization (10) leads to the following ordering of each term in (8): .

$$\frac{1}{2}(\hat{\xi}^{i_1}...\hat{\xi}^{i_k} \, \hat{p}_{j_1}...\hat{p}_{j_l} + \hat{p}_{j_1}...\hat{p}_{j_l} \, \hat{\xi}^{i_1}...\hat{\xi}^{i_k})$$

A question of fundamental importance is: which rule of ordering should be used to obtain QM of the most elementary physical system, a point-like, spinless and chargeless particle, if external gravitation is switched on in the form of nontrivial time-independent space metric?

# 4 Quantum Mechanics of Geodesic Motion in $V_n$

Having the correspondence (7) and reducing the general problem of quantization to a formulation of QM corresponding to the geodesic motion of a particle , one should only construct a quantum counterpart for the Hamilton function

$$H_0 = \frac{1}{2m}\,\omega^{ij}(\xi)\,p_i p_j \tag{11}$$

where $m$ is the mass of the particle According to the scheme given above, the problem consists in the choice of ordering of operators

$$\hat{\xi}^i,\ \hat{p}_j,\ \hat{\omega}^{ij} \stackrel{def}{=} \omega^{ij}(\hat{\xi}) = \omega^{ij}(\xi)\cdot\hat{\mathbf{1}}.$$

*Note that the metric tensor becomes now an operator* determined in fact by the von Neumann rule for quantization [15]: if $f \stackrel{Q}{\longrightarrow} \hat{f}$, then $g(f) \stackrel{Q}{\longrightarrow} g(\hat{f})$. Then, the variety of natural orderings which give $\hat{f} = \hat{f}^\dagger$ for $f(p,\xi) = \overline{f}(p,\xi)$, is exhausted by linear combinations of the Weyl and Rivier orderings considered above. However, other more exotic versions of ordering might be introduced by representing $\omega^{ij}$ as a product of other observables and taking their symmetrizations with $\hat{p}_i$ and $\hat{p}_j$, as it is done in [16], but we do not consider here these possibilities.

The Hamilton operators for the Weyl and Rivier orderings are, respectively,

$$\hat{H}_0^{(\text{Weyl})} = \frac{1}{8m}(\hat{p}_i\hat{p}_j\,\omega^{ij}(\hat{\xi}) + 2\,\hat{p}_i\,\omega^{ij}(\hat{\xi})\,\hat{p}_j + \omega^{ij}(\hat{\xi})\,\hat{p}_i\hat{p}_j) \tag{12}$$

$$\hat{H}_0^{(\text{Riv})} = \frac{1}{4m}(\hat{p}_i\hat{p}_j\,\omega^{ij}(\hat{\xi}) + \omega^{ij}(\hat{\xi})\,\hat{p}_i\hat{p}_j) \tag{13}$$

Using representation (7) for the basic operators, after some algebra, one obtains both the Hamilton operators in the form of eq.(1) with a quantum potential, respectively,

$$\hat{H}_0^{(\text{Weyl})} = -\frac{\hbar^2}{2m}\left(\Delta(\xi) + \frac{1}{2}\partial_j(\omega^{ij}\gamma_i) + \frac{1}{4}\partial_i\partial_j\omega^{ij} + \frac{1}{4}\omega^{ij}\gamma_i\gamma_j\right) \tag{14}$$

$$\hat{H}_0^{(\text{Riv})} = -\frac{\hbar^2}{2m}\left(\Delta(\xi) + \frac{1}{2}\partial_j(\omega^{ij}\gamma_i) + \frac{1}{4}\omega^{ij}\gamma_i\gamma_j\right) \tag{15}$$

$$\tag{16}$$

where $\gamma_i \stackrel{def}{=} \gamma^k{}_{ki}$, and $\gamma^k{}_{ij}$ are the Christoffel symbols for metric tensor $\omega_{ij}$ .

One sees that the quantum potentials in both the cases are noncovariant, with respect to diffeomorphisms of $V_n$, though the "kinematic" term $-(\hbar^2/2m)\Delta(\xi)$ is a scalar operator. It can be explained by that the coordinates $\xi^i$ play in the theory a two-fold role: they provide $V_n$ by a manifold structure and, at the same time, as a part of $2n$ coordinates of the phase space $X_2n$, they are classical observables. Correspondingly, $\hat{\xi}^i$ form a complete set of quantum observables, in terms of which the preparation and observation of a state of the system is performed.

The dependence of the quantum dynamics on a choice of observables (that form "the quantum statics" according to [17]) via the Hamilton operator is hidden in the standard QM in the flat space by use of Cartesian coordinates as position observables, for which $V_q = 0$. There, a use of curvilinear cordinates, e.g., the spherical ones, is done *a posteriori*, after introduction of the Hamilton operator in terms of basic observables which are firmly related to Cartesian coordinates. The curvilinear coordinates are used only as an auxiliary mathematical tool and, thus, play only the first part of the two-fold role mentioned above. The situation changes drastically in $V_n$ where one forced to use curvilinear coordinates as

basic observables from the begining, and the problem of noncovariance of the quantum potential arises inevitably.

This problem is often circimvented by taking $\hat{H}_0 = -\frac{\hbar^2}{2m}\Delta(\xi)$ as a postulate. However, it leads us out the framework of the canonical formalism, and, what is much worse, one comes then to noncovariance of the exponent term in the path integral, see [2] and Sec.5 of the present paper.

Adopting the line which we has taken, return now to the Hamilton operators 14,15. It is easy to see that no linear combination of them can produce a scalar quantum potential. However, there is a combination $\hat{H}_0^{(new)}$ which is consistent with Hamilton operator 4 in a particular class of coordinates (and, consequently, of the position observables).

To show this, consider $\hat{H}_0^{(\text{Weyl})}$ and $\hat{H}_0^{(\text{Riv})}$ in the normal quasi-Cartesian system of coordinates $\{y^a\}$, with its origin at the point $\xi^i$; indices $a, b, \ldots = 1, \ldots, n$ are further used to denote components of objects in the coordinate system $\{y^a\}$. In these coordinates, the values of $y^a$ at an arbitrary point $\xi'^i$ are $y^a = h^a s$ where $s$ is the distance along the geodesic connecting points $\xi^i$ and $\xi'^i$, and $h^a$ are components of the tangent vector to the geodesic with respect an orthonormal $n$-tuple at the origin $\xi'^i$.

In the normal quasi-Cartesian system, the metric tensor $\omega_{ab}$,

its derivatives with respect to $y^a$, and, consequently, the Christoffel symbols $\gamma^a{}_{bc}(y)$ can be expressed as a power series with coefficients that are polynomials of components of the curvature tensor

$$R^a{}_{bcd} = \partial_d \gamma^a{}_{bc} - \partial_c \gamma^a{}_{bd} + \gamma^a{}_{de}\gamma^e{}_{bc} - \gamma^a{}_{ce}\gamma^e{}_{bd}$$

and of the covariant derivatives of the tensor. Then, using the Veblen method of affine extensions [18] and the contracted Bianchi identities, one obtains the following representations for Hamilton operators:

$$\hat{H}_0^{(\text{Weyl})}(y) = -\frac{\hbar^2}{2m}\left(\Delta(y) + \frac{1}{4}\left.R\right|_{y=0} + \frac{1}{4}\left.(\partial_a R)\right|_{y=0} y^a + O(y^2)\right). \tag{17}$$

$$= -\frac{\hbar^2}{2m}\left(\Delta(\xi) + \frac{1}{4}R(\xi) + O(s^2)\right). \tag{18}$$

$$\hat{H}_0^{(\text{Riv})}(y) = -\frac{\hbar^2}{2m}\left(\Delta(y) + \frac{1}{3}\left.R\right|_{y=0} + \frac{1}{3}\left.(\partial_a R)\right|_{y=0} y^a + O(y^2)\right).$$

$$= -\frac{\hbar^2}{2m}\left(\Delta(\xi) + \frac{1}{3}R(\xi) + O(s^2)\right). \tag{19}$$

(In transitions from eq.(17) to eq.(18) and from eq.(19) to eq.(19), asymptotic relation

$$R(\xi') = R(\xi) - \partial_i R(\xi^i - \xi'^i) = R(\xi) - \partial_i R(xi)(\xi^i - \xi'^i) + O(s^2) = R(\xi) - \partial_a R(y)|_{y=0} h^a s + O(s^2)$$

is used). It is seen from here that a Hamilton operator with any coefficient of

$R(\xi)$ can be obtained by appropriate linear combination of $\hat{H}_0^{(\text{Weyl})}(y)$ and $\hat{H}_0^{(\text{Riv})}$. Our aim is to get a *local* consistency between a certain ordering rule in canonical quantization and the WKB result eq.(4) in the neighborhood of the origin $\xi'^i$, We see easily that combination

$$\hat{H}_0^{(\text{new})}(y) = 2\hat{H}_0^{(\text{Weyl})}(y) - \hat{H}_0^{(\text{Riv})}(y) \tag{20}$$
$$= -\frac{\hbar^2}{2m}\left(\Delta(\xi) + \frac{1}{6}R(\xi) + O(s^2)\right)$$

is just the Hamilton operator in the Schrödinger equation (4) in the approximation indicated there.

In terms of the basic operators $\hat{p}_j, \hat{\xi}^i$

$$\hat{H}_0^{(\text{new})} = \frac{1}{2m}\hat{p}_i \, \omega^{ij}(\hat{\xi}) \, \hat{p}_j, \tag{21}$$

that is the simplest of possible Hermitean expressions which can be constructed from $\hat{p}_i, \hat{p}_j$ and $\omega^{kl}(\hat{\xi})$.

The further logic is very simple. If an ordering rule is universal for quantization of any $f(x,p) \in \mathcal{F}$ then one should consider the combination of the Weyl and Rivier orderings in the right-hand side of eq.(20) as the general rule of ordering distinguished by the condition of local coincidence of WKB and canonical QMs. The corresponding general formula of quantization is the combination of quantizations (9) and (10)

$$f(p,\xi) \quad \xrightarrow{\mathcal{Q}} \quad (\hat{f}\psi)(\xi)$$
$$= (2\pi\hbar)^{-n}\,\omega^{-\frac{1}{4}}(\xi)\int d^n\tilde{\xi}\,d^np\,\exp\left(-\frac{i}{\hbar}(\xi^i - \tilde{\xi}^i)p_i\right)\left(2f(p,\frac{\xi+\tilde{\xi}}{2})\right)$$
$$- \frac{f(p,\xi) + f(p,\tilde{\xi})}{2}\right)\,\omega^{\frac{1}{4}}(\tilde{\xi})\psi(\tilde{\xi}). \tag{22}$$

# 5 The path integral and canonical formalism

In the path integral formalism one usually starts with the following formal expression for the propagator

$$\mathbb{K}(\xi'',t''|\xi',t') = <\xi''|e^{-\frac{i}{\hbar}(t''-t')\hat{H}_0}|\xi'> \tag{23}$$

which is proposed to be approximated as

$$\mathbb{K}(\xi'',t''|\xi',t') = \lim_{N\to\infty}\int\prod_{A=1}^{N-1}\sqrt{\omega(\xi_A)}\,d^n\xi_A\prod_{B=1}^{N-1}<\xi_A|e^{-\frac{i}{\hbar}\epsilon\hat{H}_0}|\xi_B> \tag{24}$$

8

where $\epsilon = (t'' - t')/N$ and $\xi_0 = \xi'$, $\xi_N = \xi''$. The problem is: from where does one know the Hamilton operator $\hat{H}_0$ from? At least, the following answers are possible:

1) To find it from experiments.

2) To postulate it as a differential operator in $L^2(V_n; \mathbb{C}; \sqrt{\omega}d^n\xi)$; the standard postulate is $-(\hbar/2m^2)\Delta$.

3) To quantize canonically $H_0^{(\mathrm{cl})}$, solving somehow the problem of ambiguities.

4) To conjecture the form of $\mathbb{K}(\xi'', t''|\xi', t')$ for $t'' \to t'$ and to determine $\hat{H}_0$ from the asymptotic Schrödinger equation; this is just the Pauli–DeWitt way [5].

5) To deduce the Schrödinger equation as an asymptotic of the corresponding quantum field theory, as it is done in [10].

Consider approach 2) to the path integration following [2], where the reader is referred for details. These authors, as many others, consider the general covariance of $\hat{H}_0$ as a necessary condition. (From my point of view, it is a sort of prejudice that does not take into attention the two-fold role of coordinates mentioned above.) Having taken the standard expression

$$-(\hbar/2m^2)\Delta$$

, one should express it in terms of the basic operators $\hat{p}_i$, $\hat{q}^j$. To this end, one should to take any canonically obtained Hamilton operator and to subtract the nonvanishing noncovariant quantum potential

$V_q$ corresponding to the ordering chosen. (In [2], the latter is the Weyl ordering). After that, $V_q$ will be inevitably present in the exponential of any resulting form of the path integral. For example, it reads in the lagrangean form [2]

$$\mathbb{K}(\xi'', t''|\xi', t') = \lim_{N\to\infty} \int \left(\frac{1}{2\pi i\hbar\epsilon}\right)^{\pi N/2} \prod_{A=1}^{N-1} \sqrt{\omega(\xi_A)} \, d^n\xi_A$$

$$\times \prod_{B=1}^{N-1} \frac{\omega^{1/4}(\tilde{\xi}_B)}{[\omega(\xi_B)\omega(\xi_{B-1})]^{1/4}} \exp\left[\frac{i}{\hbar}\epsilon L^{(\mathrm{eff})}\left(\tilde{\xi}_B, \frac{\xi_B - \xi_{B-1}}{\epsilon}\right)\right] \tag{25}$$

where $L^{(\mathrm{eff})}(\xi, \dot{\xi}) = L^{(\mathrm{cl})}(\xi, \dot{\xi}) - V_q(\xi)$ and $\tilde{\xi}$ denotes that the value of the function $f$ taken in a special way depending on the ordering chosen; if the Weyl one is adopted, then

$$f(\tilde{\xi}_A) \stackrel{def}{=} f\left(\frac{\xi_A - \xi_{A-1}}{2}\right))$$

; for the ordering introduced in the present paper

$$f(\tilde{\xi}_A) \stackrel{def}{=} 2f\left(\frac{\xi_A + \xi_{A-1}}{2}\right) - \frac{f(\xi_A) + f(\xi_{A-1})}{2}$$

Thus, we come to the conclusion that the direct canonical approach, that does not appeal to the general covariance, leads to $L^{(\mathrm{eff})}(\xi, \dot{\xi}) = L^{(\mathrm{cl})}(\xi, \dot{\xi})$ in eq.(25) that corresponds better to the original Feynman idea. However, a dependence of the path integral on the choice of ordering remains in the form of a rule of evalution of $f(\tilde{\xi}_A)$.

# 6 Conclusion

The approach adopted here can be considered as a sort of a "physical" experiment with different mathematical schemes of quantization. A physical system (point-like particle moving along a geodesic in $V_n$) taken as a probe is the simplest geometrically and physically meaningful one when the ambiguities of quantization clearly manifest themselves. At the same time, the system is sufficiently fundamental that the results obtained by its consideration might be used to select a preferred version of the canonical quantization, in general. The system considered reveals also a dependence of the quantum dynamics on the choice of the basic observables (on "the quantum statics") . In particular, this question seems to be important for such fundamental

problem as quantization of gravitation where a separation of the proper dynamics of the gravitational field from effects of the choice of coordintes is one of the main dificulties. In any case, our knowledge of the quantum theory would not be complete without clear realization the basic questions considered here.

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# References

[1] Berezin F A, Shubin M A 1983 *Schrödinger equation* (Moscow University Press, Moscow; in Russian)

[2] D'Olivo J C and Torres M 1989 J.Phys A.: Math., Gen. **21** 3355–3363.

[3] Bayen F , Flato M , Frionsdal C , Lichnerowicz A and Sternheimer D Ann.Phys. (N.Y.) **111**, 111 - 151.

[4] Groshe C, Pogosyan G S and Sissakian A N 1997 *Particles and Nuclei* **28** 1229–1294.

[5] DeWitt B S 1957 *Rev. Mod. Phys.* **29** 377–397.

[6] Pauli W 1951 *Feldquantisierung*, Lecture Notes, Appendix, (Zurich)

[7] Weinberg S 1975 *Gravitation and Cosmology* (New York: John Wiley and Sons).

[8] Podolsky B 1928 *Phys.Rev.* **32** 812.

[9] Śniatycki J 1980 *Geometric Quantization and Quantum Mechanics* (Springer–Verlag, New York, Heidelberg, Berlin)

[10] Tagirov E A 1999 *Class.Quantum Grav.* **16** 2165–2185.

[11] Kirillov A A 1985 in *Modern Problems of Mathematics. Fundamental Studies* **4** 141–176 (in Russian).

[12] Tagirov E A 1999 *Grav. & Cosmol.* **5** 1–10.

[13] Rivier D C 1957 Phys.Rev.**83** 862(L).

[14] Agarwal G S, Wolf E 1970 Phys.Rev.**D2** 2161 – 2186.

[15] Von Neumann J. 1955 *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton) p.313.

[16] Groshe C. 1988 Physics Letters A **128** 113 – 122.

[17] Sadbery A 1986 *Quantum Mechanics and Particles of Nature* (Cambridge: Cambridge University Press)

[18] Veblen O 1927 *Invariants of Quadratic Differential Forms* (Cambridge: Cambridge University Press).